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Some local covariant representations of the canonical anticommutation relations

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Abstract. We find some non-Fock representations of the canonical anticommutation relations in $1+1$ dimensions. These representations, obtained by a Bogoliubov transformation of the Fock annihilation and creation operators, are Poincaré covariant, have positive energy, and are labelled by two conserved charges which take on continuous values in the interval $[0, 2\pi)$.

1. Introduction

The algebraic approach to quantum field theory has been motivated by the difficulties of the standard physicists' techniques, for example with descriptions of say interacting fields or spontaneous symmetry breaking. According to the ideas of Haag and Kastler (1964) and Segal (1965; Bongaarts 1972), we should associate our various physical measurements with elements of an abstract C^*-algebra and then study the various representations of this algebra. For infinite-dimensional systems, i.e. quantum field theory, it is easy to show that there exist infinitely many unitarily inequivalent irreducible representations of our chosen algebra, all capable of describing a different 'physics'. Many of these representations, however, will not be physical in the sense of not existing in nature; indeed, to choose those of real interest, we should insist that they satisfy at least two basic conditions.

(a) The representations must be covariant under the action of the restricted Poincaré group $\mathscr{P}_+^\uparrow$, i.e. the representations must carry a strongly continuous unitary representation of $\mathscr{P}_+^\uparrow$.

(b) The representations must have positive energy, i.e. the generator of time translations must be a positive semi-definite operator on the representation space.

The various unitarily inequivalent physical representations may be labelled by their unitary invariants, i.e. their 'charges'. Superselection rules then operate between these representations: the fields intertwining these representations are supposed unobservable.

In this work we study representations of the canonical anticommutation relations over the space of solutions to the massless Dirac equation in $1+1$ dimensions, obtainable by a gauge transformation (of the second kind) in the fermion field representation, show the existence of superselection sectors, and that the representations obtained obey the above criteria. The representations we find are irreducible and quasi-free, and therefore belong to the class of covariant representations classified by Kraus and Streater (1981). Our examples are local in space–time, in the sense that they are strictly localised in the sense of Knight (1961). They are thus more in the spirit of Haag and Kastler than the examples of Kraus and Streater, which are localised in momentum space.

We follow Segal's formulation for a fermion quantum field theory: let $\mathscr{K}$ be a (one-particle) real Hilbert space and $S(f, g)$ the scalar product in $\mathscr{K}$. Let $R:\mathscr{K}\to\mathfrak{A}_0$ be a real-linear injection into the abstract complex algebra $\mathfrak{A}_0$, generated by $R(\mathscr{K})$, such that

$$[R(f), R(g)]_+ = R(f)R(g)+R(g)R(f) = S(f, g)\mathbb{1}$$

where $\mathbb{1}$ is the identity in $\mathfrak{A}_0$. Let $*$ be an involution in $\mathfrak{A}_0$ such that the $R(f)$ are all self-adjoint, and furnish $\mathfrak{A}_0$ with the (unique) norm such that the completion $\mathfrak{A}$ of $\mathfrak{A}_0$ in this norm is a C^*-algebra: the Clifford algebra over $(\mathscr{K}, S)$.

A representation of the canonical anticommutation relations is a real-linear $*$-homomorphism from $\mathfrak{A}$ into the set of all bounded operators acting on some Hilbert space $\mathscr{H}$—known as the representation space. We seek the existence of a vacuum (Fock) representation, which apart from satisfying the conditions of Poincaré covariance and positivity of energy, should contain the Poincaré invariant cyclic vector Ω—the vacuum.

Symmetries are introduced into the theory as $*$-automorphisms of the algebra; as in Kraus and Streater (1981) we limit our study to those induced by the one-particle transformations $R(f)\to R(Vf)$, where V is a real orthogonal operator on $\mathscr{K}$. V need not be unitary, though; this possibility may lead us to a non-spatial automorphism and hence enable us to construct examples of non-Fock representations starting from the vacuum representation. The dynamics of these representations will be completely determined by the dynamics of the vacuum representation: if τ_t is the time evolution automorphism, then the 'time evolution' of an automorphism σ is given by $\tau_t \cdot \sigma \cdot \tau_t^{-1}$.

To make $\mathscr{K}$ into a complex Hilbert space we have to introduce a complex structure J, such that when we multiply $f\in\mathscr{K}$ by a complex number $a+\mathrm{i}b$, we get

$$(a+\mathrm{i}b)f = af + bJf.$$

Given a complex structure J we may define a new scalar product $\langle\cdot,\cdot\rangle$:

$$\langle f, g\rangle = S(f, g) + \mathrm{i}S(Jf, g)$$

which makes $\mathscr{K}$ into a complex Hilbert space. The Fock representation of is defined relative to a J. For example, let $\mathscr{K}$ be the complex Hilbert space of square integrable functions on the real line, complete with respect to the inner product

$$\langle f, g\rangle = \int \mathrm{d}x\, \overline{f(x)}g(x).$$

Then $S(f, g) = \mathrm{Re}\langle f, g\rangle$. $\mathscr{K}$ carries the complex structure $J = \mathrm{i}$, and we define the field representation in terms of its annihilation and creation operators, $\psi(f)$, $\psi^*(f)$, with respect to this complex structure: let

$$\psi(f) = (1/\sqrt{2})(R(f)+\mathrm{i}R(\mathrm{i}f)) \qquad \psi^*(f) = (1/\sqrt{2})(R(f)-\mathrm{i}R(\mathrm{i}f)).$$

These operators act on the 'field space', given by

$$= \mathbb{C}\oplus\mathscr{K}\oplus(\mathscr{K}\oplus\mathscr{K})_{\mathrm{A}}\oplus(\mathscr{K}\otimes\mathscr{K}\otimes\mathscr{K})_{\mathrm{A}}\oplus\ldots$$

$$= \mathscr{K}_0\oplus\mathscr{K}_1\oplus\mathscr{K}_2\oplus\mathscr{K}_3\oplus\ldots$$

such that

$$\psi^*(f): \Psi_n(f_1, \ldots, f_n) \to \Psi_{n+1}(f, f_1, \ldots, f_n)$$

$$\psi(f): \Psi_n(f_1, \ldots, f_n) \to \sum_j (-1)^j \langle f_j, f\rangle \Psi_{n-1}(f_1, \ldots, \hat{f}_j, \ldots, f_n)$$

$$\psi(f)\Psi_0 = 0$$

where $\Psi_n \in \mathscr{K}_n$, etc, and $\Psi_0 \in \mathbb{C}$ is the normalised cyclic vector for this representation, which we call the field representation.

To introduce time evolution, we make the functions $f \in \mathscr{K}$ time dependent, according to the massless Dirac equation; $\mathscr{K}$ is then the Hilbert space of functions for which

$$f(t, x) = e^{iHt} f(0, x) = e^{iHt} f(x)$$

is also square integrable in x, where $H = -i\gamma_5 \partial/\partial x$, and $\gamma_5 = \gamma_0\gamma_1$. We choose the γ-matrix representation

$$\gamma_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \gamma_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

H, the generator of time translations, is then self-adjoint on $(\mathscr{K}, \langle \cdot, \cdot \rangle)$; however, it is not bounded below. The field representation is not a physical representation therefore, though it is Poincaré covariant (Streater and Wightman 1978): for each $(a, \Lambda) \in \mathscr{P}_+^\uparrow$ where

$$a = (a^0, a^1) \qquad \Lambda(\lambda) = \begin{bmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{bmatrix}$$

the action of the Poincaré automorphism $\tau_{(a,\Lambda)}$ is given by

$$\tau_{(a,\Lambda)}: \psi(f) \to \exp(i\gamma_5\lambda/2)\psi(f_{(a,\Lambda)})$$

and

$$f_{(a,\Lambda)}(t, x) = f[\Lambda^{-1}(t - a^0, x - a^1)].$$

We now wish to find the physical vacuum representation of this theory. As we have seen, the natural complex structure in $\mathscr{K}$ is not suitable for its definition, as the Hamiltonian is not bounded below. We can, however, split it up uniquely into positive and negative parts: there exist orthogonal projections P_+, P_-, such that

$$HP_+ = P_+H = H_+ \geq 0 \qquad HP_- = P_-H = H_- \leq 0.$$

Let $J = i(P_+ - P_-)$ be a new complex structure in $\mathscr{K}$; the new inner product will be

$$\langle f, g\rangle_J = S(f, g) + iS(f, g) = \mathrm{Re}\langle f, g\rangle + i\,\mathrm{Im}\langle (P_+ - P_-)f, g\rangle.$$

As H commutes with J, the action of the time evolution operator $U(t) = e^{iHt}$ will again be unitary on $\mathscr{K}$, with respect to the new scalar product $\langle \cdot, \cdot \rangle_J$. We could write

$$U(t) = e^{J\tilde{H}t}$$

where

$$\tilde{H} = -JiH = (P_+ - P_-)H = H_+ - H_- \geq 0$$

i.e. $U(t)$ has a positive generator with respect to the new Hilbert space structure. We may define new annihilation and creation operators with respect to J: let

$$a(f) = (1/\sqrt{2})(R(f) + iR(Jf)) \qquad a^*(f) = (1/\sqrt{2})(R(f) - iR(Jf)).$$

These annihilation and creation operators belong to the vacuum representation, which is certainly Poincaré covariant, and as shown above, has positive energy. Finally we express the fermion field in terms of these operators:

$$\psi(f) = a(P_+f) + a^*(P_-f).$$

As P_+, P_- are unique orthogonal projections, the one-particle Hilbert space $\mathcal{K}$ divides itself naturally into two components:

$$\mathcal{K}_+ = \{f \in \mathcal{K}, P_+f \neq 0\} \qquad \mathcal{K}_- = \{f \in \mathcal{K}, P_-f \neq 0\}.$$

$\mathcal{K}_+$, $\mathcal{K}_-$ are usually referred to as the (one) particle and (one) antiparticle spaces respectively. In particular, $a(P_+f)$, $a^*(P_-f)$ are the particle annihilation and anti-particle creation operators.

2. Gauge transformations and the new representations

Consider the action of the operator $V(\alpha)$ on the field representation space given by

$$V(\alpha): \Psi_n(f_1, \ldots, f_n) \to \Psi_n(\mathrm{e}^{\mathrm{i}\alpha}f_1, \ldots, \mathrm{e}^{\mathrm{i}\alpha}f_n)$$

where α is real. The automorphism of the algebra $\mathfrak{A}$ which this induces is known as a gauge transformation, and α may be a function of the (space) point x. Though this action is unitary in the field representation, it need not necessarily be so in the vacuum representation if we make $\alpha(x)$ a suitable function. Streater and Wilde (1970; Bonnard and Streater 1977) found examples of non-Fock representations of the canonical commutation relations, obtainable by a related automorphism, by choosing $\alpha(x)$ in the form of a smooth step function, i.e. $\alpha \in C^\infty(\mathbb{R})$, and $\mathrm{d}\alpha/\mathrm{d}x \in C_0^\infty(\mathbb{R})$. Following their lead, let $\alpha(x)$, $\alpha_5(x)$ be of the form above, and consider the set G of automorphisms, σ_{α,α_5} of $\mathfrak{A}$, defined at time zero by its one-particle action

$$\sigma_{\alpha,\alpha_5}: \psi(f) \to V^*(\alpha+\gamma_5\alpha_5)\psi(f)V(\alpha+\gamma_5\alpha_5) = \psi\{\exp[\mathrm{i}(\alpha+\gamma_5\alpha_5)]f\}$$

where V^* is the inverse of V. G forms a group with multiplication rule

$$\sigma_{\alpha,\alpha_5}\cdot\sigma_{\beta,\beta_5} = \sigma_{(\alpha+\beta),(\alpha_5+\beta_5)}.$$

To simplify calculation, we shall consider the group $\mathrm{G}_0 = \mathrm{G}/(\mathrm{U}(1)\times\mathrm{U}(1))$, i.e. the group of automorphisms isomorphic to those for which $\alpha(-\infty) = \alpha_5(-\infty) = 0$.

In the physical vacuum representation, the action of G_0 will cause the mixture of particle annihilation and antiparticle creation operators, and vice versa; in terms of Fourier transforms (denoted by $\hat{}$)

$$\hat{\psi}^*\{\exp[\mathrm{i}(\alpha+\gamma_5\alpha_5)]*\hat{f}\} = \hat{a}\{P_+\exp[\mathrm{i}(\alpha+\gamma_5\alpha_5)]*\hat{f}\} + \hat{a}^*\{P_-\exp[\mathrm{i}(\alpha+\gamma_5\alpha_5)]*\hat{f}\}$$

where the $*$ denotes convolution, so the automorphism σ_{α,α_5} induces the Bogoliubov transformation

$$\hat{a}(p_+\hat{f}) \to \hat{a}(V_{++}\hat{f}) + \hat{a}^*(V_{-+}\hat{f}) \qquad \hat{a}^*(P_-\hat{f}) \to \hat{a}^*(V_{--}\hat{f}) + \hat{a}(V_{+-}\hat{f})$$

where

$$V_{\varepsilon\varepsilon'}\hat{f} = P_\varepsilon(\widehat{\exp[\mathrm{i}(\alpha+\gamma_5\alpha_5)]}*P_{\varepsilon'}\hat{f})$$

and $\varepsilon, \varepsilon' = +$ or $-$. A necessary and sufficient condition (Berezin 1966, Ruijsenaars 1977) for this transformation to be implemented by means of unitary operators in

Fock space is if the off-diagonal parts, i.e. V_{-+} and V_{+-}, belong to the class of Hilbert–Schmidt kernels of operators on $\mathscr{K}$.

We have chosen to work with Fourier transforms since P_+, P_- then take on a particularly simple form:

$$P_+ = \begin{bmatrix} \theta(p) & 0 \\ 0 & \theta(-p) \end{bmatrix} \qquad P_- = \begin{bmatrix} \theta(-p) & 0 \\ 0 & \theta(p) \end{bmatrix}.$$

Then

$$V_{-+}f = \begin{bmatrix} \theta(-p) & 0 \\ 0 & \theta(p) \end{bmatrix} \int \mathrm{d}q \begin{bmatrix} \widehat{\exp[\mathrm{i}(\alpha+\alpha_5)]}(p-q) & 0 \\ 0 & \widehat{\exp[\mathrm{i}(\alpha-\alpha_5)]}(p-q) \end{bmatrix}$$

$$\times \begin{bmatrix} \theta(q) & 0 \\ 0 & \theta(-q) \end{bmatrix} \begin{bmatrix} \hat{f}_1(q) \\ \hat{f}_2(q) \end{bmatrix}$$

so the kernel

$$V_{-+}(p,q) = \begin{bmatrix} \theta(-p)\theta(q)\,\widehat{\exp[\mathrm{i}(\alpha+\alpha_5)]}(p-q) & 0 \\ 0 & \theta(p)\theta(-q)\,\widehat{\exp[\mathrm{i}(\alpha-\alpha_5)]}(p-q) \end{bmatrix}.$$

To determine the Hilbert–Schmidt norms of V_{-+}, V_{+-}, we note that the functions $\exp[\mathrm{i}(\alpha \pm \alpha_5)]$ are C^∞, equal to 1 to the left of some point, and equal to a constant $\exp[\mathrm{i}(\alpha(\infty) \pm \alpha_5(\infty))] = h$ to the right of another point. We may therefore write either one as $\exp[\mathrm{i}(\alpha \pm \alpha_5)] = 1 + \int_{-\infty}^{x} \psi \, \mathrm{d}x$ where $\psi \in \mathscr{D}$, and $\int_{-\infty}^{\infty} \psi(x)\,\mathrm{d}x = h - 1 = (2\pi)^{1/2}\tilde{\psi}(0)$.

The projection P_- commutes with the unit operator, and so $P_+ 1 P_- = 0$. The Fourier transform of $\int_{-\infty}^{X} \psi(x)\,\mathrm{d}x$ is $\tilde{\psi}(p)/(p + \mathrm{i}\varepsilon)$. Hence $v = (V_{+-})_{11}$ has kernel $\theta(-p)[\tilde{\psi}(p-q)/(p-q+\mathrm{i}\varepsilon)]\theta(q)$. The Hilbert–Schmidt norm of this is

$$\|v\|_2^2 = \int_{-\infty}^{\infty} \mathrm{d}p \int_{-\infty}^{\infty} \mathrm{d}q \, \theta(-p)\theta(q) \left| \frac{\tilde{\psi}(p-q)}{p-q+\mathrm{i}\varepsilon} \right|^2;$$

put $s = p - q$, $t = p + q$, to get

$$\|v\|_2^2 = \int_{-\infty}^{\infty} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}t \, \theta(-t-s)\theta(-s+t) \left| \frac{\tilde{\psi}(s)}{s^2} \right|^2 = \int_{-\infty}^{0} s \, \mathrm{d}s \left| \frac{\tilde{\psi}(s)}{s} \right|^2.$$

Now $\psi \in \mathscr{S}$; this integral is finite if and only if $\tilde{\psi}(0) = 0$, i.e. $h = 1$, giving $\alpha(\infty) + \alpha_5(\infty) = 2\pi n$. Similarly the norm of $(V_{-+})_{22}$ is finite if and only if $\alpha(\infty) - \alpha_5(\infty) = 2\pi m$. Summarising this result:

Theorem 1. Let σ_{α,α_5}: $\psi(f) \to \psi\{\exp[\mathrm{i}(\alpha + \gamma_5\alpha_5)]f\}$ be a gauge transformation of the massless fermion field operator where $\alpha(x)$, $\alpha_5(x)$ are C^∞ functions with first derivatives of compact support. Then the automorphism of the fermion algebra generated by this transformation is unitarily implementable in the physical vacuum representation if and only if

$$\left.\begin{aligned} \alpha(\infty) + \alpha_5(\infty) &= \alpha(-\infty) + \alpha_5(-\infty) \\ \alpha(\infty) - \alpha_5(\infty) &= \alpha(-\infty) - \alpha_5(-\infty) \end{aligned}\right\} \text{modulo } 2\pi.$$

The non-unitarily implementable gauge transformations $\{\sigma_{\alpha,\alpha_5}$: $\alpha(\infty) \pm \alpha_5(\infty) \neq 0$, modulo $2\pi\}$ give rise to inequivalent representations $\pi_{\alpha,\alpha_5}(\mathfrak{A}) = \pi_0 \cdot \sigma_{\alpha,\alpha_5}(\mathfrak{A})$ of the canonical anticommutation relations, with creation and annihilation operators

$a_{\alpha,\alpha_5}{}^*(f)$, $a_{\alpha,\alpha_5}(f)$ given by the Bogoliubov transformation. Note that the non-implementability of these automorphisms was dependent only on the value of the functions $\alpha \pm \alpha_5$ at $x = \infty$, but not on their particular form. This suggests that two different sets of functions α, α_5; β, β_5 will give us unitarily equivalent representations if they have the same behaviour at infinity, modulo 2π. This is indeed the case.

Theorem 2. Let $\sigma_{\alpha,\alpha_5}: \psi(f) \to \psi\{\exp[i(\alpha + \gamma_5\alpha_5)]f\}$, $\sigma_{\beta,\beta_5}: \psi(f) \to \psi\{\exp[i(\beta + \gamma_5\beta_5)]f\}$, be two gauge transformations of the free fermion field, where α, α_5; β, β_5 are functions of the form discussed above; then the two representations $\pi_{\alpha,\alpha_5}(\mathfrak{A}) = \pi_0 \cdot \sigma_{\alpha,\alpha_5}(\mathfrak{A})$; $\pi_{\beta,\beta_5}(\mathfrak{A}) = \pi_0 \cdot \sigma_{\beta,\beta_5}(\mathfrak{A})$, where π_0 is the physical vacuum representation, are unitarily equivalent if and only if

$$\left.\begin{aligned}\alpha(\infty) + \alpha_5(\infty) &= \beta(\infty) + \beta_5(\infty)\\ \alpha(\infty) - \alpha_5(\infty) &= \beta(\infty) - \beta_5(\infty)\end{aligned}\right\}\text{modulo } 2\pi.$$

Proof. $\pi_{\alpha,\alpha_5}(\mathfrak{A})$, $\pi_{\beta,\beta_5}(\mathfrak{A})$ will be equivalent if and only if there exists a unitary operator U, say, such that for all $A \in \mathfrak{A}$,

$$\pi_{\alpha,\alpha_5}(A) = U\pi_{\beta,\beta_5}(A)U^{-1}$$

i.e. if

$$\pi_0 \cdot \sigma_{\alpha,\alpha_5}(A) = U(\pi_0 \cdot \sigma_{\beta,\beta_5}(A))U^{-1}$$

i.e.

$$\pi_0 \cdot \sigma_{\alpha,\alpha_5} \cdot \sigma_{\beta,\beta_5}^{-1}(A) = U\pi_0(A)U^{-1}$$

that is, if and only if $\sigma_{\alpha,\alpha_5} \cdot \sigma_{\beta,\beta_5}^{-1}$ is unitarily implementable in Fock representations, π_0. Now (up to a phase)

$$\begin{aligned}(\sigma_{\alpha,\alpha_5} \cdot \sigma_{\beta,\beta_5}^{-1}\psi)(f) &= \psi\{\exp[i(\alpha + \gamma_5\alpha_5)]\exp[-i(\beta + \gamma_5\beta_5)]f\}\\ &= \psi[\![\exp\{i[\alpha - \beta + \gamma_5(\alpha_5 - \beta_5)]\}f]\!].\end{aligned}$$

However, by theorem 1, this transformation will be unitary if and only if

$$\left.\begin{aligned}\alpha(\infty) - \beta(\infty) + \alpha_5(\infty) - \beta_5(\infty) &= 0\\ \alpha(\infty) - \beta(\infty) - (\alpha_5(\infty) - \beta_5(\infty)) &= 0\end{aligned}\right\}\text{modulo } 2\pi.$$

We see that the representations divide themselves up into equivalence classes labelled by their unitary invariants, the charges

$$\left.\begin{aligned}\alpha_+ &= \alpha(\infty) + \alpha_5(\infty)\\ \alpha_- &= \alpha(\infty) - \alpha_5(\infty)\end{aligned}\right\}\text{modulo } 2\pi.$$

In particular, the equivalence class of the physical vacuum representation has charges $\alpha_+ = \alpha_- = 0$.

3. Poincaré covariance and positivity of the energy

We have made our one-particle space $\mathscr{K}$ time dependent in the sense that each $f \in \mathscr{K}$ is a solution to the massless Dirac equation: $-i\partial\!\!\!/ f = 0$. Every $f \in \mathscr{K}$ is therefore a spinor whose upper component is right moving, and lower left moving: at any time t,

$$f(t, x)=\begin{bmatrix} f_1(0, x-t) \\ f_2(0, x+t) \end{bmatrix}.$$

The upper and lower components of the gauge transformed field $\psi\{\exp[\mathrm{i}(\alpha+\gamma_5\alpha_5)]f\}$ will evolve in time in the same way, i.e.

$$(\exp[\mathrm{i}(\alpha+\gamma_5\alpha_5)]f)(t, x)=\begin{bmatrix} \exp[\mathrm{i}\alpha_1(x-t)]f_1(0, x-t) \\ \exp[\mathrm{i}\alpha_2(x+t)]f_2(0, x+t) \end{bmatrix}$$

where $\alpha_1(x)=\alpha(x)+\alpha_5(x)$, $\alpha_2(x)=\alpha(x)-\alpha_5(x)$. Under a Poincaré transformation $(a, \Lambda)\in\mathscr{P}_+^\uparrow$

$$x-t\to \mathrm{e}^{-\lambda}[x-a^1-(t-a^0)] \qquad x+t\to\mathrm{e}^{\lambda}(x-a^1+t-a^0).$$

To show that the representations obtained are Poincaré covariant we must first show that σ_{α,α_5} and $\tau_{(a,\Lambda)}\cdot\sigma_{\alpha,\alpha_5}$ define unitarily equivalent representations of the type concerned, and then that the action of the automorphism is strongly continuous in the representation space. By the same arguments as in the proof of theorem 2, the first part is equivalent to showing that $\sigma_{\alpha,\alpha_5}\cdot\tau_{(a,\Lambda)}\cdot\sigma^{-1}_{\alpha,\alpha_5}$ is unitarily implementable in the physical vacuum representation. This is the map

$$\psi(f)\to\exp(\mathrm{i}\gamma_5\lambda/2)\psi\{\exp[\mathrm{i}(\alpha+\gamma_5\alpha_5)](\exp[-\mathrm{i}(\alpha+\gamma_5\alpha_5)]f)_{(a,\Lambda)}\}.$$

Now

$$\begin{aligned}
&\exp[\mathrm{i}(\alpha+\gamma_5\alpha_5)](\exp[\mathrm{i}(\alpha+\gamma_5\alpha_5)]f)_{(a,\Lambda)}\\
&\quad=\exp[\mathrm{i}(\alpha+\gamma_5\alpha_5)]\begin{bmatrix} \exp[-\mathrm{i}\alpha_1(x-t)]f_1(t, x) \\ \exp[-\mathrm{i}\alpha_2(x+t)]f_2(t, x) \end{bmatrix}_{(a,\Lambda)}\\
&\quad=\exp[\mathrm{i}(\alpha+\gamma_5\alpha_5)]\begin{bmatrix} \exp(\mathrm{i}\alpha_1\{\mathrm{e}^{-\lambda}[x-a^1-(t-a^0)]\})f_{1_{(a,\Lambda)}}(t, x) \\ \exp\{-\mathrm{i}\alpha_2[\mathrm{e}^{\lambda}(x-a^1+t-a^0)]\}f_{2_{(a,\Lambda)}}(t, x) \end{bmatrix}\\
&\quad=\exp\left(\mathrm{i}\begin{bmatrix} \alpha_1(x-t)-\alpha_1[\mathrm{e}^{-\lambda}(x-a^1-(t-a^0)] & \mathbf{0} \\ \mathbf{0} & \alpha_2(x+t)-\alpha_2[\mathrm{e}^{\lambda}(x-a^1+t-a^0)] \end{bmatrix}\right)\\
&\quad\times f_{(a,\Lambda)}(t, x).
\end{aligned}$$

This transformation is unitarily implementable in the physical vacuum representation, for in the limit $x\to\infty$

$$\alpha_1(x-t)-\alpha_1\{\mathrm{e}^{-\lambda}[x-a^1-(t-a^0)]\}\to 0 \qquad \alpha_1(x+t)-\alpha_2[\mathrm{e}^{\lambda}(x-a^1+t-a^0)]\to 0$$

so fulfilling the conditions of theorem 1, and the map $f\to f_{(a,\Lambda)}$ is certainly unitary as the physical vacuum representation is covariant. Moreover, as α_1, α_2 are both C^∞ functions, the action of $\tau_{(a,\Lambda)}$ will be strongly continuous in the representation space associated with π_{α_+,α_-}, since continuous actions on the one-particle space induce (norm)-continuous actions on $\mathfrak{A}$ (Kraus and Streater 1981). We have now proved theorem 3.

Theorem 3. The representations $\pi_{\alpha_+,\alpha_-}(\mathfrak{A})$ defined as above are Poincaré covariant.

Theorem 4. The representations so defined have positive energy.

Proof. To prove that the energy, H', is bounded below we write it as a Wick-ordered creation–annihilation form, and apply the method of Glimm (1967).

$$H' = \int \mathrm{d}x : \psi'^*(x)\gamma^5 \frac{\partial}{\partial x}\psi'(x):$$

where

$$\psi'(x) = \exp\{\mathrm{i}[\alpha(x) + \gamma_5\alpha_5(x)]\}\psi(x)$$

with the Wick ordering being in the physical vacuum state, and $\psi(x)$ being in the relativistic vacuum representation of the field. Now

$$\psi(x) = \frac{1}{\sqrt{2\pi}}\int \mathrm{d}p\, \mathrm{e}^{\mathrm{i}px}\left(\begin{bmatrix}\theta(p)\\ \theta(-p)\end{bmatrix}a(p) + \begin{bmatrix}\theta(-p)\\ \theta(p)\end{bmatrix}b^*(-p)\right)$$

where $a(p)$, $b^*(-p)$ are the particle annihilation and antiparticle creation operators respectively. From this we see that

$$H' = H + \int \mathrm{d}x : \psi^*(x)\gamma^5 \frac{\partial(\alpha + \gamma_5\alpha_5)}{\partial x}\psi(x):$$

$$= H + \int \mathrm{d}x : \psi_1^*(x)\frac{\partial(\alpha + \alpha_5)}{\partial x}\psi_1(x): + \int \mathrm{d}x : \psi_2^*(x)\frac{\partial(\alpha - \alpha_5)}{\partial x}\psi_2(x):.$$

The first term is of course bounded below as it is the energy of the vacuum representation. We consider the second term, the energy of the right-going waves; the third term, the energy of the left-going waves, is treated similarly.

Let $\alpha_1(x) = \alpha(x) + \alpha_5(x)$; then

$$\int \mathrm{d}x : \psi_1^*(x)\frac{\partial\alpha_1}{\partial x}\psi_1(x):$$

$$= \int \mathrm{d}p\, \mathrm{d}q : [\theta(p)a^*(p) + \theta(-p)b(-p)]\widehat{\alpha_1'}(p-q) \times [\theta(q)a(q) + \theta(-q)b^*(-q)]:$$

$$= \int \mathrm{d}p\, \mathrm{d}q\, \theta(p)\theta(q)[\widehat{\alpha_1'}(p-q)a^*(p)a(q) + \widehat{\alpha_1'}(p+q)a^*(p)b^*(q) + \widehat{\alpha_1'}(-p-q)b(p)a(q) - \widehat{\alpha_1'}(-p+q)b^*(q)b(p)].$$

In the infrared region, say $0 \leqslant p \leqslant \kappa$, the four kernels of these operators (call a typical one r) obey $\||p|^{-\tau/2}r\|_2 < \infty$, $0 < \tau < 1$. The four operators represented by the integrals over $0 \leqslant p \leqslant \kappa$ are therefore infinitely small relative to the free energy (Glimm 1967, theorem 2.4.3). The vacuum polarisation terms have kernels of the Hilbert–Schmidt class over $\kappa \leqslant p \leqslant \infty$, since

$$\int_\kappa^\infty \mathrm{d}p \int_0^\infty \mathrm{d}p |\widehat{\alpha_1'}(p-q)|^2 = \int_\kappa^\infty \mathrm{d}q |\widehat{\alpha_1'}(q)|^2 \int_\kappa^q \mathrm{d}p < \infty.$$

Therefore they are infinitely small relative to the number operator (Glimm 1967, corollary to theorem 2.4.2) and therefore to H if $\kappa > 1$. The remaining terms

$$A = \int_\kappa^\infty \mathrm{d}p \int_0^\infty \mathrm{d}q\, \widehat{\alpha'_1}(p-q)a^*(p)a(q) - \int_\kappa^\infty \mathrm{d}p \int_0^\infty \mathrm{d}q\, \widehat{\alpha'_1}(-p+q)b^*(q)b(p)$$

are second quantisations of the (bounded) multiplication operators $\alpha'_1(\pm x)$ with the projection operators $\theta(p-\kappa)$, $\theta(q)$, and so have norm $\leq \sup_x |\alpha'(x)| = M$, say. If $\kappa > 4M$, this term is dominated by $\frac{1}{2}H$, i.e. $\frac{1}{2}H + A \geq 0$. Collecting these results we see that H' is bounded below.

4. Final remarks

It will be shown in a further paper (Gallone *et al* 1983) that the equal-time currents $j^\mu = \bar{\psi}\gamma^\mu\psi$ and $j^{5\mu} = \bar{\psi}\gamma^5\gamma^\mu\psi$ make sense in the representations π_{α,α_5} when smeared with test functions β, β_5 in $\mathscr{D}(\mathbb{R})$. They are the generators, respectively, of the local gauge transformations $e^{i\alpha}$ and $\exp(i\gamma^5\alpha^5)$. Whereas these automorphisms commute, j^0 and j^{50} do not commute, but satisfy the canonical commutation relations. The abelian gauge group is thus represented by a Weyl system. The representation of this Weyl system in π_{α,α_5} is unitarily equivalent to the charge $(\alpha(\infty), \alpha_5(\infty))$ representation of the free boson field (Streater and Wilde 1970), when the former is restricted to the cyclic space generated from the transformed vacuum state. Moreover, the representations of the canonical commutation relations are inequivalent even when the charges are equivalent, modulo 2π. This is possible because the currents are reducibly represented in the Hilbert space of the canonical anticommutation relations, so one representation can contain many inequivalent Weyl systems. For more details, see Gallone *et al* (1983).

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